Universität Augsburg

Institut für Mathematik

Olga Ruff, Friedrich Pukelsheim

A Stochastic Synopsis of Binary Voting Rules

Preprint Nr. 32/2009 — 18. November 2009

Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

Impressum:

Herausgeber:
Institut für Mathematik
Universität Augsburg
86135 Augsburg
`http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml`

ViSdP:
Friedrich Pukelsheim
Institut für Mathematik
Universität Augsburg
86135 Augsburg

A STOCHASTIC SYNOPSIS OF BINARY VOTING RULES

Olga Ruff and Friedrich Pukelsheim

Institute for Mathematics, University of Augsburg, Germany

Binary voting rules are discussed with a view towards probability theory and statistics. For selfdual and permutationally invariant distributions, the majority rule and the unanimity rule are shown to bound the mean success margin of any other voting rule. The Penrose/Banzhaf model uses the uniform distribution over all possible decisions. Bloc voting rules lead to product distributions beyond uniformity. The Shapley/Shubik approach entails correlated voting behavior.

1. Introduction. The discussion of voting rules and the measurement of power originates from, and usually is oriented towards, game theory. Von Neumann/Morgenstern (1944) laid the foundation for the game-theoretic approach, and Shapley (1962), Coleman (1971), Owen (1971), and Dubey (1975) followed their lead. Felsenthal/Machover (1998) present a detailed overview of the subject, including a critical assessment of concepts and methods. The present paper is an attempt to provide an alternative approach, motivated by a decision-theoretic view used in statistics.

With our statistical background we occasionally felt some irritation that, when authors make probabilistic statements, it remains unclear to which probability space they refer to. Is it the set of voters? Is it the set of permutations into which the voters may be aligned? Is it the space of decisions? Is it the space of all agendas to be treated during the voting process? Our paper grew out of an attempt to extract just one reference space and see how much of the current theory can be developed within the space chosen.

Our decision space Ω_N consists of vectors a, called decisions. For every voter j in a finite assembly N, a decision a records whether j votes Yea ($a_j = 1$) or Nay ($a_j = 0$). This decision space figures prominently also in the Felsenthal/Machover (1998) monograph, and in current research literature such as Laruelle/Valenciano (2004, 2005).

Section 2 introduces a voting rule W_N as a set of decisions that forms a monotone, nonempty, and proper subset of the decision space Ω_N. Important events are $C_j(W_N)$, consisting of the decisions where voter j may exert critical decisiveness. An important function is the success margin $\alpha_{W_N}(a)$, the difference between the number of voters for whom the decision a is a success and the number of those for whom it is a failure.

Section 3 turns to general probability assumptions. Two properties become vital, selfduality of a distribution P, and permutational invariance. For such distributions, Theorem 2 proves that the mean success margin of a voting rule W_N is bounded from above by the mean success margin of the majority rule, while it is bounded from below by the mean success margin of the unanimity rule. This result is well-known for the Penrose/Banzhaf model. Yet, our theorem shows that it is due to general probabilistic properties, rather than being owed to the particular structure of the uniform distribution.

Section 4 turns to the Penrose/Banzhaf model. It is now easy to see that the sensitivity of a voting rule coincides with its mean success margin. In Section 5 we overview other power indices, by identifying them as conditional probabilities or conditional expectations in the Penrose/Banzhaf model.

Section 6 focuses on bloc voting rules. Theorem 4 derives a product formula for the influence probability of voter j, in the presence of a prespecified partitioning into blocs. The formula splits into the impact of voter j in his or her bloc L, and the impact of bloc L relative to the other blocs within the partitioning. The formula generalizes, and compactifies, a result due to Felsenthal/Machover (2002). Partitionings of the assembly N into blocs are also used by Laruelle/Valenciano (2004) and, for the investigation of list apportionments in proportional representation systems, by Leutgäb/Pukelsheim (2009).

Section 7 merges the Shapley/Shubik indices into the present approach. As pointed out by Dubey/Shapley (1979), the Shapley/Shubik model may be based on a two-stage usage of uniform distributions. Section 8 concludes the paper with some final remarks.

2. Decision space and voting rules. Let N be an *assembly*, a finite set, of n *voters*. A *decision* is a vector $a = (a_j)_{j\in N}$ with binary components, $a_j := 1$ in case voter $j \in N$ is a *Yea-voter*, or $a_j := 0$ in case j is a *Nay-voter*. Altogether the decisions form the *decision space*

$$\Omega_N := \{0,1\}^N.$$

Let $0_N := (0,\ldots,0)$ denote the zero vector and $\mathit{1}_N := (1,\ldots,1)$ the unity vector, each with n components. For a given decision $a \in \Omega_N$ the component-wise partial ordering $\le$ of vectors induces the interval region $[a, \mathit{1}_N] := \{b \in \Omega_N \,:\, a \le b \le \mathit{1}_N\}$. A binary *voting rule* (also known as a simple voting game) is a subset $W_N \subseteq \Omega_N$ enjoying three properties:

$$\begin{aligned}
&(1)\quad [a, \mathit{1}_N] \subseteq W_N \qquad \text{for all } a \in W_N,\\
&(2)\quad \mathit{1}_N \in W_N,\\
&(3)\quad 0_N \notin W_N.
\end{aligned}$$

The decisions in W_N are called *positive* (also known as winning coalitions, winning configurations), those in the complement $W_N^c := \Omega_N \setminus W_N$ *negative*. In view of the monotonicity property (1), properties (2) and (3) mean that the positive decisions form a nonempty and proper subset of the decision space, $\emptyset \neq W_N \neq \Omega_N$.

Voting rules are usually described in a form staying closer to the way a committee operates. Let $w = (w_j)_{j\in N} \in (0,\infty)^N$ be a *voting weight vector*. The scalar product $a'w := \sum_{j\in N} a_j w_j$ designates the *weight of the decision* a, the sum of the voting weights of the Yea-voters in a. The maximal possible weight is the weight of the full assembly N, and is abbreviated by $w_+ := \mathit{1}_N{}'w$.

By definition, the *weighted voting rule* $W_N(q;w)$ contains the decisions for which the weight exceeds the pre-specified relative *quota* $q \in [0,1)$:

$$W_N(q;w) := \Big\{ a \in \Omega_N \ : \ a'w > qw_+ \Big\}.$$

In the symmetric case all voters possess the same voting weight, turning the weight vector into $w = \lambda \mathit{1}_N$ for some $\lambda > 0$. The most prominent examples are the *unanimity rule* U_N and the straight *majority rule* M_N:

$$\begin{aligned} U_N &:= W_N(1-1/n; \mathit{1}_N) = \{\mathit{1}_N\}, \\ M_N &:= W_N\,(1/2; \mathit{1}_N) = \{a \in \Omega_N \ : \ a_+ > n/2\}. \end{aligned}$$

The Euclidean unit vector $e_j := (0,\ldots,0,1,0,\ldots,0)$ captures the decision where j is the sole Yea-voter. The decisions in which the vote of j becomes *critical* (decisive) are assembled in the event

$$C_j(W_N) := \Big\{ a \in \Omega_N \ : \ \big(\, a \in W_N^c, \; a + e_j \in W_N \, \big) \text{ or } \big(\, a \in W_N, \; a - e_j \in W_N^c \, \big) \Big\}.$$

That is, a voter is either *entry-critical* (critical outside a decision) when leaving the Nay-voters and joining the Yea-voters turns a negative decision into a positive deicision. Or the voter is *exit-critical* (critical in a decision) when switching from the Yea-voters to the Nay-voters turns a positive decision into the negative.

The critical event $C_j(W_N)$ is understood better by concentrating, not on voter j, but on the competitors $N \setminus \{j\}$. To this end let $\Pi_{N\setminus\{j\}}$ be the projection of the decision space $\Omega_N = \{0,1\}^N$ onto $\Omega_{N\setminus\{j\}} = \{0,1\}^{N\setminus\{j\}}$, the $(n-1)$-dimensional marginal space leaving out voter j. This is one instance—out of more to follow—where it proves helpful to use sets as subscripts, such as the assembly and its subsets, and not their cardinalities.

Given a decision $b \in \Omega_{N\setminus\{j\}}$ without voter j, we denote the decision when j joins in with a Yea by $(b;1)$, and when j votes Nea by $(b;0)$. The set of decisions where the vote of j is critical may then be rewritten as

$$C_j(W_N) := \Big\{ a \in \Omega_N \ : \ \big(\Pi_{N\setminus\{j\}}(a);1\big) \in W_N, \quad \big(\Pi_{N\setminus\{j\}}(a);0\big) \in W_N^c \Big\}.$$

Theorem 1. *Let W_N be a voting rule for an assembly N. Let $D_j(W_N) := \Pi_{N\setminus\{j\}}\big(C_j(W_N)\big)$ denote the image of the critical event $C_j(W_N)$ for voter $j \in N$ under the projection $\Pi_{N\setminus\{j\}}$. Then $C_j(W_N)$ is the pre-image of $D_j(W_N)$:*

$$C_j(W_N) = \Pi^{-1}_{N\setminus\{j\}}\big(D_j(W_N)\big).$$

Proof. A vector $a \in \Omega_N$ is mapped to the image $b := \Pi_{N\setminus\{j\}}(a) \in \Omega_{N\setminus\{j\}}$, with components $b_i = a_i$ for all $i \neq j$. The vector $b \in \Omega_{N\setminus\{j\}}$ has two pre-images, $(b;0)$ and $(b;1)$. Hence we obtain $D_j(W_N) = \big\{b \in \Omega_{N\setminus\{j\}} \,:\, (b;0) \in W_N^c,\ (b;1) \in W_N\big\}$. Evidently the pre-image of $D_j(W_N)$ reproduces the event $C_j(W_N)$. □

A decision $a \in \Omega_N$ is said to be a *success* for voter j provided it is positive and j is a Yea-voter ($a \in W_N$, $a_j = 1$), or it is negative and j is an Nay-voter ($a \in W_N^c$, $a_j = 0$). A positive decision is taken to be a *failure* for a Nay-voter, as is a negative decision for a Yea-voter. The notion of success is emphasized by Laruelle/Valenciano (2005) as a property capturing an aspect somewhat complementary to criticality.

The difference between the number of voters for which a decision $a \in \Omega_N$ is a success, and the number of the voters for which it appears to be a failure, defines the *success margin* $\alpha_{W_N}(a)$ of the voting rule W_N:

$$\alpha_{W_N}(a) := \begin{cases} a_+ - (n - a_+) = 2a_+ - n & \text{in case } a \in W_N, \\ (n - a_+) - a_+ = n - 2a_+ & \text{in case } a \in W_N^c. \end{cases}$$

A particular emphasis is placed on positive decisions $a \in W_N$ appearing to be a failure to a majority of voters. For such decisions the success margin is negative, whence its negative part represents the *majority deficit*, $\delta_{W_N} := \alpha^-_{W_N} = (|\alpha_{W_N}| - \alpha_{W_N})/2$.

A few additional notions will be useful. A decision $\mathit{1}_N - a$ is called the *dual decision of* $a \in \Omega_N$. The *dual decision rule of* W_N is defined to be $W_N^* := \{\mathit{1}_N - a \in \Omega_N \,:\, a \in W_N^c\}$. Denoting the cardinality of W_N by ω, we obtain $\#W_N^* = \#W_N^c = 2^n - \omega$. A voter j with $C_j(W_N) = \emptyset$ is never critical, and is called a *dummy*. In the voting rule $W_N = [e_j, \mathit{1}_N]$ voter j, being able to determine the outcome of the decision, is called a *dictator*.

With these set theoretic preparations we now evaluate the events of interest by means of appropriate probability distributions.

3. Distributional assumptions. The aim is to equip the decision space Ω_N with probability distributions P permitting a meaningful *a priori* analysis of decision rules W_N.

The share of all positive decisions $P(W_N)$ is called the *decision-making ability* of the decision rule W_N under P (also known as P-efficiency). The *influence probability* under P (also known as swing probability), of voter j in the decision rule W_N, is defined to be $P\big(C_j(W_N)\big)$. The sum of all influence probabilities, $\Sigma_P(W_N) := \sum_{j\in N} P\big(C_j(W_N)\big)$, is termed the *$P$-sensitivity* of the decision rule W_N.

The critical events $C_j(W_N)$, $j \in N$, generally neither cover the decision space Ω_N, nor turn out to be pairwise disjoint. Hence only in special circumstances will the P-sensitivity be equal to one. However, the P-sensitivity can be used to normalize the influence probabilities into $P\big(C_j(W_N)\big)/\Sigma_P(W_N)$. The normalized influence probabilities are called the *share of power of voter j* under P. The power shares form a probability distribution for the assembly N, preserving for any two voters $i \neq j$ the ratio of their influence probabilities, $P\big(C_i(W_N)\big)/P\big(C_j(W_N)\big)$.

Some structural properties of P become essential. A distribution P is said to be *selfdual* when $P(\{a\}) = P(\{\mathit{1}_N - a\})$ holds for all $a \in \Omega_N$. Selfduality means that the probability for a decision on a bill is just the same as the probability for the dual decision on the negation of that bill.

A distribution P is called *permutationally invariant* when $P \circ \sigma^{-1} = P$ holds for all bijections (one-to-one and onto mappings) $\sigma : N \to N$. To see the effect of the property, we decompose the decision space into the sets of decisions with a fixed number k of Yea-voters:

$$\Omega_N = \biguplus_{k=0}^{n} \left\{ {N \atop k} \right\}, \qquad \left\{ {N \atop k} \right\} := \Big\{ a \in \Omega_N \;:\; a_+ = k \Big\}.$$

The subset $\left\{ {N \atop k} \right\}$ has cardinality $\binom{n}{k}$. Within such a subset, a permutationally invariant distribution behaves like a uniform distribution:

$$P(\{a\}) = P\left(\left\{ {N \atop k} \right\} \right) \Big/ \binom{n}{k} \qquad \text{for all } a \in \left\{ {N \atop k} \right\}.$$

Theorem 2. *Let W_N be a decision rule for an assembly N.*

(i) *The success margin and the majority deficit of W_N are related to the success margin of the majority rule M_N through $\alpha_{W_N} = \alpha_{M_N} - 2\delta_{W_N} \le \alpha_{M_N}$. In particular, every distribution P fulfills*

$$\mathrm{E}_P[\alpha_{W_N}] \le \mathrm{E}_P[\alpha_{M_N}].$$

(ii) *Every selfdual distribution P fulfills*

$$\mathrm{E}_P[\alpha_{W_N}] = 2 \sum_{a \in W_N} (2a_+ - n) P(\{a\}) = 2 \sum_{k=1}^{n} (2k-n) P\left(W_N \cap \left\{ \begin{matrix} N \\ k \end{matrix} \right\} \right).$$

(iii) *Every selfdual and permutationally invariant distribution P fulfills*

$$\mathrm{E}_P[\alpha_{W_N}] \ge \mathrm{E}_P[\alpha_{U_N}] = 2nP(\{\mathit{1}_N\}).$$

Proof. (i) The absolute value of any success margin is equal to the success margin of the majority rule, since $|\alpha_{W_N}(a)| = |2a_+ - n| = \alpha_{M_N}(a)$. The assertions follow from $\delta_{W_N} = (\alpha_{M_N} - \alpha_{W_N})/2$, and $\alpha_{W_N} \le |\alpha_{W_N}|$.

(ii) For $a \in \Omega_N$ we define the indicator function

$$\mathbb{1}\{a \in W_N\} = \begin{cases} 1 & \text{in case } a \in W_N, \\ 0 & \text{in case } a \in W_N^c. \end{cases}$$

Thus the success margin turns into $\alpha_{W_N}(a) = (2 \cdot \mathbb{1}\{a \in W_N\} - 1)(2a_+ - n)$. It follows that $\mathrm{E}_P[\alpha_{W_N}] = 2\sum_{a\in W_N}(2a_+ - n)P(\{a\}) - \sum_{a \in \Omega_N}(2a_+ - n)P(\{a\})$. The last sum vanishes due to the selfduality of P:

$$\sum_{a \in \Omega_N} \big(a_+ - (\mathit{1}_N - a)_+\big) P(\{a\}) = \sum_{a \in \Omega_N} a_+ P(\{a\}) - \sum_{a \in \Omega_N} (\mathit{1}_N - a)_+ P(\{\mathit{1}_N - a\}) = 0.$$

The second equality of the assertion rearranges the sum according to the count of the Yea-voters, observing $0_N \notin W_N$. The final sum has at most n terms, while the penultimate sum may have up to $2^n - 1$ terms.

(iii) Since $\mathit{1}_N \in W_N$, part (ii) yields $\mathrm{E}_P[\alpha_{W_N}] = 2nP(\{\mathit{1}_N\}) + 2\sum_{k=1}^{n-1}(2k-n) \times P\left(W_N \cap \left\{ \begin{smallmatrix} N \\ k \end{smallmatrix} \right\}\right)$. It remains to show that the second sum is nonnegative. To this end we subdivide the range of summation into two regions of equal cardinality, $1 \leq k < n/2$ and $n/2 < k \leq n-1$. Applying permutational invariance and selfduality, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-1}(2k-n)P\left(W_N \cap \left\{ {N \atop k} \right\}\right) \\
&\quad = \sum_{1\leq k<n/2}(n-2k)\left(P\left(W_N \cap \left\{ {N \atop n-k} \right\}\right) - P\left(W_N \cap \left\{ {N \atop k} \right\}\right)\right) \\
&\quad = \sum_{1\leq k<n/2} \frac{(n-2k)P\left(\left\{ {N \atop k} \right\}\right)}{\binom{n/2}{k}}\left(\#\left(W_N \cap \left\{ {N \atop n-k} \right\}\right) - \#\left(W_N \cap \left\{ {N \atop k} \right\}\right)\right).
\end{aligned}$$

Since a voting rule W_N is monotone, by its defining property (1), any set of decisions with many Yea-voters $(n-k)$ outnumbers a set of decisions with only a few Yea-voters (k), for all $1 \leq k < n/2$. This proves all terms of the final sum to be nonnegative. □

4. The Penrose/Banzhaf model. The simplest distributional model is the *Penrose/ Banzhaf distribution* P_N, the uniform distribution on the space Ω_N:

$$P_N(\{a\}) := \frac{1}{\#\Omega_N} = \frac{1}{2^n} \quad \text{for all } a \in \Omega_N.$$

The Penrose/Banzhaf distribution makes the voting behavior of all voters $j \in N$ stochastically independent and identically distributed, $P_N = \bigotimes_{j\in N} P_{\{j\}}$. The one-dimensional marginal distributions are $P_{\{j\}} = \text{Bernoulli}(1/2)$. The distribution P_N is selfdual and permutationally invariant. The Penrose/Banzhaf decision-making ability (also known as Coleman's *power of a collectivity to act*) is $P_N(W_N) = \omega/2^n$.

For a given voting rule W_N, the *Penrose/Banzhaf influence probability of the voter* $j \in N$ evaluates to

$$P_N\big(C_j(W_N)\big) = P_N \circ \Pi^{-1}_{N\setminus\{j\}}\big(D_j(W_N)\big) = P_{N\setminus\{j\}}\big(D_j(W_N)\big) = \frac{\eta_j}{2^{n-1}},$$

where $\eta_j := \#D_j(W_N)$ denotes the *swing score* (also known as Banzhaf score) of voter j. The following result is well-known, see Theorem 3.3.5 in Felsenthal/Machover (1998).

THEOREM 3. *In the Penrose/Banzhaf model, the P-sensitivity of every voting rule W_N coincides with its mean success margin:*

$$\Sigma_{P_N}(W_N) = \frac{1}{2^{n-1}} \sum_{a \in W_N} (2a_+ - n) = \mathrm{E}_{P_N}[\alpha_{W_N}].$$

PROOF. By definition, $\Sigma_{P_N}(W_N) = \eta_+/2^{n-1}$. We set $\omega_j := \#\{a \in W_N \,:\, a_j = 1\}$. If voter j changes camps, then the positive decisions for which j is critical become negative and drop out: $\#\{a \in W_N \,:\, a_j = 0\} = \omega_j - \eta_j$. We obtain $\omega = 2\omega_j - \eta_j$, that is, $\eta_j = 2\omega_j - \omega$. Substituting $\omega_j = \sum_{a \in W_N} \mathbb{1}\{a_j = 1\}$ and summing over j, we get $\eta_+ = 2\left(\sum_{a \in W_N} \sum_{j \in N} \mathbb{1}\{a_j = 1\}\right) - n\omega = \sum_{a \in W_N}(2a_+ - n)$. This proves the first equality. The second equation follows from Theorem 2. □

Theorems 2 and 3 imply that the unanimity rule U_N and the majority rule M_N provide bounds for the Penrose/Banzhaf sensitivity of a voting rule W_N:

$$\frac{n}{2^{n-1}} = \Sigma_{P_N}(U_N) \;\le\; \Sigma_{P_N}(W_N) \;\le\; \Sigma_{P_N}(M_N) = \frac{n}{2^{n-1}} \binom{n-1}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2n}{\pi}}.$$

5. Conditional power indices in the Penrose/Banzhaf model. Other power indices emerge from the Penrose/Banzhaf model as conditional probabilities, or as conditional expectations. For an appraisal of these indices see Felsenthal/Machover (1998).

5.1. Conditioning on the set of positive decisions. A first group of power indices conditions on the voting rule W_N itself, thereby emphasizing its interpretation as the set of all positive decisions.

The Penrose/Banzhaf probability of the critical event $C_j(W_N)$ given the positive decisions W_N is better known as *Coleman's power to prevent action*:

$$\mathrm{E}_{P_N}\left[\mathbb{1}\big\{a \in C_j(W_N)\big\} \,\Big|\, W_N\right] = P_N\left(C_j(W_N) \,\Big|\, W_N\right) = \frac{P_N\big(C_j(W_N) \cap W_N\big)}{P_N(W_N)}$$
$$= \frac{\eta_j/2^n}{\omega/2^n} = \frac{\eta_j}{\omega}.$$

The penultimate equality uses that the number of positive decisions for which voter j is exit-critical is $\#\big(C_j(W_N) \cap W_N\big) = \#\big(C_j(W_N) \cap W_N^c\big) = \#C_j\,(W_N)\,/2 = \eta_j$.

Coleman's power to initiate action is given by $P_N\big(C_j(W_N)\,|\,W_N^c\big) = \eta_j/(2^n - \omega)$. Here the conditioning event is provided by the negative decisions W_N^c, and η_j represents the number of negative decisions where voter j is entry-critical. The harmonic mean of the two Coleman indices reproduces the Penrose/Banzhaf influence probability $P_N\big(C_j(W_N)\big)$. This reproduction property applies in every probability space (Ω_N, P) where the relation $P\big(C_j(W_N) \cap W_N\big) = P\big(C_j(W_N) \cap W_N^c\big)$ holds true.

Generally the two Coleman indices do not sum to unity. In either case normalization reproduces the Penrose/Banzhaf influence probabilities.

An alternative idea is that in case of an increasing number of critical Yea-voters they should be assigned decreasing pay-offs. This reasoning originates from a game-theoretic approach, the winning coalition of Yea-voters having to share a fixed prize. For a decision $a \in \Omega_N$ we define the vector $\gamma(a)$ to indicate whether voter j is exit-critical ($\gamma_j(a) := 1$), or not ($\gamma_j(a) := 0$). Hence, the component sum $\gamma_+(a)$ indicates the number of exit-critical Yea-voters. The *Burgin/Shapley index* is defined as

$$\begin{aligned} \mathrm{E}_{P_N}\left[\frac{1}{\gamma_+(a)}\mathbb{1}\big\{a \in C_j(W_N)\big\} \,\Big|\, W_N\right] &= \sum_{a \in C_j(W_N)} \frac{1}{\gamma_+(a)} P_N\left(\{a\} \,\Big|\, W_N\right) \\ &= \sum_{a \in C_j(W_N) \cap W_N} \frac{1}{\gamma_+(a)} \frac{1/2^n}{\omega/2^n} = \frac{1}{\omega} \sum_{a \in C_j(W_N) \cap W_N} \frac{1}{\gamma_+(a)}. \end{aligned}$$

The normalized versions of the Burgin/Shapley indices are called *Johnston indices*.

5.2. Conditioning on the set of minimal-positive decisions. A second group of indices arises when the conditioning event is taken to be the set of *minimal-positive decisions*, that is, decisions wherein every Yea-voter is exit-critical:

$$W_N^{\min} := \Big\{a \in W_N \;:\; a - \mathbb{1}\{a_j = 1\}e_j \in W_N^c \quad \text{for all } j \in N\Big\}.$$

The interval regions that are induced by the minimal-positive decisions characterize the decision rule: $W_N = \bigcup_{a \in W_N^{\min}} [a, \mathit{1}_N]$. Kirsch/Langner (2009) make do with minimal-positive decisions to calculate influence probabilities.

The indices corresponding to Coleman's power to prevent action are

$$\mathrm{E}_{P_N}\left[\mathbb{1}\big\{a \in C_j(W_N)\big\} \,\Big|\, W_N^{\min}\right] = \frac{\#\big(C_j(W_N) \cap W_N^{\min}\big)}{\#W_N^{\min}}.$$

Normalization yields the *Holler/Packel public good indices.*

In minimal-positive decisions every Yea-voter is exit-critical, $\gamma(a) = a$. The indices that run parallel to the Burgin/Shapley indices are the *Deegan/Packel indices*

$$\mathrm{E}_{P_N}\left[\frac{1}{a_+}\mathbb{1}\{a \in C_j(W_N)\} \,\middle|\, W_N^{\min}\right] = \frac{1}{\#W_N^{\min}} \sum_{a \in C_j(W_N) \cap W_N^{\min}} \frac{1}{a_+}.$$

Their total happens to be equal to unity, since for all $a \in W_N^{\min}$ we get $\sum_{j \in N} \mathbb{1}\{a \in C_j(W_N)\} = a_+$, and $\sum_{j \in N} \mathrm{E}_{P_N}\left[\frac{1}{a_+}\mathbb{1}\{a \in C_j(W_N)\} \,\middle|\, W_N^{\min}\right] = 1$.

Although the Penrose/Banzhaf uniform distribution is the most prominent model, bloc voting rules give rise to other interesting distributions.

6. Bloc voting rules. We assume that an assembly N is given, together with its decision space Ω_N and a decision rule W_N. A *partitioning* $\mathcal{L}$ of the assembly N is a decomposition of N into pairwise disjoint subsets. Its subsets $K \in \mathcal{L}$ are called *blocs*.

The smallest partitioning is $\{N\}$, embracing just the single bloc N. The largest partitioning is $\{\{j\} : j \in N\}$, featuring only *trivial*—that is, one-element—blocs $\{j\}$. These two configurations are extreme and only of theoretical interest. Practical examples use partitionings $\mathcal{L}$ consisting of more than one and less than n blocs. In Exhibit 1 we partition the former EEC into four blocs, $\mathcal{L} = \{\{\mathrm{DE}\}, \{\mathrm{IT}\}, \{\mathrm{FR}\}, \{\mathrm{NL, BE, LU}\}\}$. The big members stay alone, while the Benelux states join into a three-member bloc.

The assembly N is a disjoint union of the blocs $K \in \mathcal{L}$, and the decision space is a Cartesian product of the decision spaces of the blocs:

$$N = \biguplus_{K \in \mathcal{L}} K, \qquad \Omega_N = \prod_{K \in \mathcal{L}} \Omega_K.$$

A decision in Ω_N now is a block vector $a = (a_K)_{K \in \mathcal{L}}$, with components $a_K := (a_j)_{j \in K}$.

Given a bloc $K \in \mathcal{L}$, we consider the assembly K and its corresponding decision space Ω_K. We assume that every bloc is given an *internal* decision rule W_K. The final decision, in the grand assembly N, is preceded by internal bloc decisions. If in bloc K the internal decision a_K is positive, then all members of the bloc vote Yea in the final decision. If a_K is negative, all of them vote Nay. The contribution of bloc K to the final decision therefore is $\mathbb{1}\{a_K \in W_K\}\mathit{1}_K$, namely $\mathit{1}_K$ in case $a_K \in W_K$, and 0_K otherwise. This leads to the formal definition of the *bloc voting rule* to be

$$W_N|(W_K)_{K \in \mathcal{L}} := \left\{(a_K)_{K \in \mathcal{L}} \in \Omega_N \;:\; \left(\mathbb{1}\{a_K \in W_K\}\mathit{1}_K\right)_{K \in \mathcal{L}} \in W_N\right\} \subseteq \Omega_N.$$

Theorem 4 treats the partitioning $\mathcal{L}$ as another assembly, as in Felsenthal/Machover (2002) and Laruelle/Valenciano (2004). The decision space $\Omega_{\mathcal{L}}$ is equipped with a decision rule $W_{\mathcal{L}}$ that is induced by the decision rule W_N.

THEOREM 4. *Let $\mathcal{L}$ be a partitioning of the assembly N. With voting rules W_N for N and W_K for the blocs $K \in \mathcal{L}$, we introduce*

$$W_{\mathcal{L}} := \{b \in \Omega_{\mathcal{L}} \ : \ (b_K \mathit{1}_K)_{K \in \mathcal{L}} \in W_N\}, \qquad Q_{\mathcal{L}} := \bigotimes_{K \in \mathcal{L}} \mathrm{Bernoulli}\big(P_K(W_K)\big).$$

Then we have, for every bloc $L \in \mathcal{L}$ and for all voters $j \in L$:

$$P_N\Big(C_j\left(W_N \big| (W_K)_{K \in \mathcal{L}}\right)\Big) = P_L\big(C_j(W_L)\big)\, Q_{\mathcal{L}}\big(C_L(W_{\mathcal{L}})\big).$$

PROOF. For every decision a in Ω_N, the indicators $b_K(a_K) := \mathbb{1}\{a_K \in W_K\}$ induce the decision $b(a) := \big(b_K(a_K)\big)_{K \in \mathcal{L}}$ in $\Omega_{\mathcal{L}}$. A voter $j \in L$ is critical in Ω_N, with respect to the bloc voting rule $W_N|(W_K)_{K \in \mathcal{L}}$, if and only if j is critical in W_L and the bloc L is critical in $W_{\mathcal{L}}$:

$$C_j\left(W_N \big| (W_K)_{K \in \mathcal{L}}\right) = \Big\{a \in \Omega_N \ : \ a_L \in C_j(W_L), \ \ b(a) \in C_L(W_{\mathcal{L}})\Big\}.$$

By Theorem 1, the event $C_L(W_{\mathcal{L}}) = \Pi^{-1}_{\mathcal{L} \setminus \{L\}}\big(D_L(W_{\mathcal{L}})\big)$ depends on the blocs in $\mathcal{L} \setminus \{L\}$, only. Since the distribution P_N is a product, $P_N = \bigotimes_{K \in \mathcal{L}} P_K$, we obtain

$$P_N\Big(C_j\left(W_N \big| (W_K)_{K \in \mathcal{L}}\right)\Big) = P_L\Big(C_j(W_L)\Big)\, P_{N \setminus L}\Big(\big\{(b_K)_{K \in \mathcal{L} \setminus \{L\}} \in D_L(W_{\mathcal{L}})\big\}\Big).$$

In the last factor we re-introduce the marginal space Ω_L:

$$P_{N \setminus L}\Big(\big\{(b_K)_{K \in \mathcal{L} \setminus \{L\}} \in D_L(W_{\mathcal{L}})\big\}\Big) = P_N\Big(\big\{b \in C_L(W_{\mathcal{L}})\big\}\Big) = P_N \circ b^{-1}\big(C_L(W_{\mathcal{L}})\big).$$

The distribution of the random vector $b = (b_K)_{K \in \mathcal{L}}$ under P_N turns out to be

$$P_N \circ b^{-1} = \bigotimes_{K \in \mathcal{L}} P_K \circ b_K^{-1} = \bigotimes_{K \in \mathcal{L}} \mathrm{Bernoulli}(p_K),$$

with $p_K := P_K\big(\{b_K = 1\}\big) = P_K\big(\{a_K \in \Omega_K \ : \ a_K \in W_K\}\big) = P_K(W_K)$. This yields the distribution $Q_{\mathcal{L}}$ as claimed in the assertion. □

Trivial blocs $K = \{j\}$ do not really contribute anything novel to the product formula. Indeed, the sole decision rule for them is $W_{\{j\}} = \{1\}$. We obtain $P_{\{j\}}\big(C_j(W_{\{j\}}\big) = 1$, whence the first factor in the product formula equals unity. Moreover, they enter into the distribution $Q_{\mathcal{L}}$ as a Bernoulli(1/2) component, since $P_{\{j\}}(W_{\{j\}}) = 1/2$. For this reason trivial blocs are often omitted when listing the members of a partitioning.

In other words, if in a partitioning $\mathcal{L}$ a voter j stands alone, then the behavior of the trivial bloc $\{j\}$ in the partitioning assembly $\mathcal{L}$ is identical with the behavior of the voter j in the original assembly N, with a common probability $1/2$ of being a Yea-voter. Thus a voter who stays back as a one-element bloc remains passive, and falls victim to the competing blocs of the partitioning $\mathcal{L}$.

In Theorem 4 the distribution $Q_{\mathcal{L}}$ is a product of Bernoulli distributions. But a Yea in bloc K has probability $P_K(W_K)$, and a Nay has probability $1 - P_K(W_K)$. In general, it is no longer necessarily the case that a Yea emerges with probability $1/2$. For example in Exhibit 1, the Benelux bloc votes Yea under the unanimity rule with probability $1/8$.

More than that, other instances give rise to distributions with correlated components. An example is the Shapley/Shubik distribution.

7. The Shapley/Shubik model. The *Shapley/Shubik distribution* S_N is based on a superposition of uniform distributions, see Dubey/Shapley (1979). Every subset $\left\{ {N \atop k} \right\}$, $k = 0, \ldots, n$, is assigned the same probability $1/(n+1)$. Conditionally on such a subset, its $\binom{n}{k}$ decisions are also assumed to be uniformly distributed:

$$S_N(\{a\}) := \frac{1}{(n+1)\binom{n}{a_+}} \qquad \text{for all } a \in \Omega_N.$$

It is easy to verify that the Shapley/Shubik distribution is selfdual and permutationally invariant. Moreover, Theorem 5 shows that the family of Shapley/Shubik distributions is projectively consistent with respect to its marginal distributions.

THEOREM 5. *For all voters $j \in N$ we have $S_N \circ \Pi^{-1}_{N\setminus\{j\}} = S_{N\setminus\{j\}}$.*

PROOF. For $b \in \Omega_{N\setminus\{j\}}$ we have $\Pi^{-1}_{N\setminus\{j\}}\{b\} = \{(b;0),(b;1)\}$. Set $k = b_+$. The identities $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ justify the assertion:

$$\begin{aligned} S_N\Big(\{(b;0),(b;1)\}\Big) &= \frac{1}{(n+1)\binom{n}{k}} + \frac{1}{(n+1)\binom{n}{k+1}} = \frac{\binom{n}{k+1} + \binom{n}{k}}{(n+1)\binom{n}{k}\binom{n}{k+1}} \\ &= \frac{1}{(k+1)\binom{n}{k+1}} = \frac{1}{n\binom{n-1}{k}} = S_{N\setminus\{j\}}(\{b\}). \end{aligned}$$

□

The Shapley/Shubik influence probability of voter j becomes

$$S_N\big(C_j(W_N)\big) = S_{N\setminus\{j\}}\big(D_j(W_N)\big) = \sum_{k=0}^{n-1} S_{N\setminus\{j\}}\left(\left\{ {N\setminus\{j\} \atop k} \right\} \cap D_j(W_N)\right).$$

Since in the probability space $(\Omega_{N\setminus\{j\}}, S_{N\setminus\{j\}})$ a uniform distribution rules on the subsets $\left\{ {N\setminus\{j\} \atop k} \right\}$, we introduce the counting variables

$$\eta_j(k) := \#\left(\left\{ {N\setminus\{j\} \atop k} \right\} \cap D_j(W_N)\right) =: s_{k+1,j},$$

for all $k \in \{0,\ldots,n-1\}$ and $j \in N$. The number s_{ij} counts the decisions consisting of i Yea-voters (including j) and featuring voter j as exit-critical. Altogether they form the $\{1,\ldots,n\} \times N$ *swing matrix* $s = (\!(s_{ij})\!)$. Therefore, we obtain

$$S_N\big(C_j(W_N)\big) = \sum_{k=0}^{n-1} \frac{\eta_j(k)}{n\binom{n-1}{k}} = \frac{1}{n!}\sum_{k=0}^{n-1} k!(n-1-k)!\,\eta_j(k) = \frac{1}{n!}\sum_{i=1}^{n}(i-1)!(n-i)!\,s_{ij}.$$

Theorem 6 states that the Shapley/Shubik influence probabilities always sum to unity. Hence the notion of sensitivity is superfluous, in the Shapley/Shubik model.

THEOREM 6. *In the Shapley/Shubik model every decision rule W_N has S_N-sensitivity equal to unity:*

$$\sum_{j\in N} S_N\big(C_j(W_N)\big) = 1.$$

PROOF. The assertion is entirely of combinatorial nature: $\sum_{i=1}^{n}(i-1)!(n-i)!s_{i+} = n!$. We show that the left hand side counts all permutations of n voters, as does the right hand side. On the left hand side the counting is carried out in a way that is dictated by the problem. Without loss of generality we assume that the assembly is arranged in the form $N = \{1,\ldots,n\}$. Let $\sigma(1),\ldots,\sigma(n)$ be an arbitrary permutation of the voters. We count the cases where the sole Yea-voters are $\sigma(1),\ldots,\sigma(i)$ and where voter $\sigma(i)$ is exit-critical:

$$e_{\sigma(1)} + \cdots + e_{\sigma(i-1)} + e_{\sigma(i)} \in W_N, \qquad e_{\sigma(1)} + \cdots + e_{\sigma(i-1)} \in W_N^c.$$

Voter $j := \sigma(i)$ maintains the exit-critical role in all permutations that originate from rearranging the predecessors $\sigma(1),\ldots,\sigma(i-1)$, as well as rearranging the successors $\sigma(i+1),\ldots,\sigma(n)$. This generates $(i-1)!(n-i)!$ permutations. Finally, the number $s_{i+} := \sum_{j\in N} s_{ij}$ is the count of how often voter j takes the position of the exit-critical voter $\sigma(i)$. $\square$

Theorem 6 entails the rather strange consequence that, for weighted voting rules $W_N(q;\lambda \mathit{1}_N)$ where all voters enjoy the same voting weight λ, the Shapley/Shubik influence probabilities of all voters are equal to $1/n$. They do not depend on the quota q, and therefore the Shapley/Shubik model is incapable of distinguishing the unanimity rule U_N (with quota $q = 1 - 1/n$), from the majority rule M_N (with quota $q = 1/2$).

In the Shapley/Shubik model the mean success margin does not coincide with the P-sensitivity, but provides extra information. Theorem 2 provides the bounds

$$\frac{2n}{n+1} = \mathrm{E}_{S_N}[\alpha_{U_N}] \;\le\; \mathrm{E}_{S_N}[\alpha_{W_N}] \;\le\; \mathrm{E}_{S_N}[\alpha_{M_N}] = \frac{n+1}{2} - \begin{cases} \dfrac{1}{2(n+1)} & \text{in case } n \text{ even,} \\ 0 & \text{in case } n \text{ odd.} \end{cases}$$

The Shapley/Shubik model assigns weights to the subsets $\left\{ {N \atop k} \right\}$ that differ from those in the Penrose/Banzhaf model:

$$S_N\left(\left\{ {N \atop k} \right\}\right) = \frac{1}{n+1} \neq \frac{1}{2^n}\binom{n}{k} = P_N\left(\left\{ {N \atop k} \right\}\right).$$

Nevertheless, within any such subset the conditional probabilities are the same. For all decisions $a \in \left\{ {N \atop k} \right\}$ we have $S_N\left(\{a\} \mid \left\{ {N \atop k} \right\}\right) = 1/\binom{n}{k} = P_N\left(\{a\} \mid \left\{ {N \atop k} \right\}\right)$.

The Shapley/Shubik model has marginal distributions $S_{\{j\}} = \mathrm{Bernoulli}(1/2)$, for all voters $j \in N$, as has the Penrose/Banzhaf model. However, any two voters are stochastically dependent in their behavior, $\mathrm{Cov}_{S_N}[a_i, a_j] = \mathrm{Cov}_{S_{\{i,j\}}}[a_i, a_j] = 1/12$. Their correlation coefficient turns out to be $(1/12)/(1/4) = 1/3$.

The positive correlation becomes visible also in the conditional probabilities

$$S_N\left(\{(b;1)\} \mid \{(b;0),(b;1)\}\right) = \frac{S_N\big(\{(b;1)\}\big)}{S_{N\setminus\{j\}}\big(\{b\}\big)} = \frac{b_+ + 1}{n+1}.$$

In the Shapley/Shubik model voter j turns into a Yea-voter with a likelihood that increases with the number of Yea-voters (b_+) surrounding j. This is reminiscent of the *accessus* procedure in clerical elections. The accession of minority voters to the majority may ease the way to a two-thirds winning coalition, see Colomer/McLean (1998).

8. Conclusion. In this paper we leave the common game theoretic ground and instead rely on tools from probability theory and statistics. The use of the single decision space Ω_N, consisting of all possible decisions in an assembly N, provides us with an appropriate framework to embed many prominent power measures known in the literature. This leads to a general theory in which power measures arise from appropriate distributional assumptions.

We have shown in Section 3 that the upper und lower bounds of the expected success margin apply to all distributions that are selfdual and permutationally invariant, rather than being restricted to special assumptions such as the Penrose/Banzhaf model. Hence we are able to calculate the mean majority deficit in these cases, in particular for the Shapley/Shubik model.

Bloc voting rules, which we have studied in Section 6, provide us with an entirely new class of interesting distributions and yield a generalization to Felsenthal/Machover (2002). We allow blocs of any sizes and arbitrary internal voting rules. The example of a Benelux bloc in the former EEC illustrates the different power distributions among the six states when the majority rule, or the unanimity rule, is used within the Benelux bloc.

Finally, we would like to remark that our approach extends to ternary voting rules where abstentions are allowed, see Käufl/Ruff/Pukelsheim (2009). Contingent on the probability $t \in [0, 1)$ for abstaining, that paper developes formulas embracing the results of the present paper as the case $t = 0$.

REFERENCES

COLOMER, J.M. / MCLEAN, I. (1998). Electing popes: Approval balloting and qualified-majority rule. *Journal of Interdisciplinary History* **29** 1–22.

COLEMAN, J.S. (1971). Control of collectivities and the power of a collectivity to act. In: Lieberman, B. (Editor), *Social Choice.* New York, 269–300.

DUBEY, P. (1975). On the uniqueness of the Shapley value. *International Journal of Game Theory* **4** 131-140.

DUBEY, P. / SHAPLEY, L.S. (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research* **4** 99–131.

FELSENTHAL, D.S. / MACHOVER, M. (1998). *The Measurement of Voting Power – Theory and Practice, Problems and Paradoxes.* Chaltenham UK. s FELSENTHAL, D.S. / MACHOVER, M. (2002). Annexations and allicances: When are blocs advantageous a prior? *Social Choice and Welfare* **19** 295–312.

KÄUFL, A. / RUFF, O. / PUKELSHEIM, F. (2009). A stochastic approach to ternary voting rules. Forthcoming.

KIRSCH, W. / LANGNER, J. (2009). Power indices and minimal winning coalitions. *Social Choice and Welfare,* DOI 10.1007/s00355-009-0387-3.

LARUELLE, A. / VALENCIANO, F. (2004). On the meaning of Owen-Banzhaf coalitional value in voting situations. *Theory and Decisions* **56** 113–123.

LARUELLE, A. / VALENCIANO, F. (2005). Assessing success and decisiveness in voting situations. *Social Choice and Welfare* **24** 171-197.

LEUTGÄB, P. / PUKELSHEIM, F. (2009). List apparentements in local elections: A lottery. In: *Festschrift Hannu Nurmi.* München.

OWEN, G. (1971). Political games. *Naval Research Logistics Quarterly* **18** 345-355.

SHAPLEY, L.S. (1962). Simple games: An outline of the descriptive theory. *Behavioral Science* **7** 59–66.

VON NEUMANN, J. / MORGENSTERN, O. (1944). *Game Theory and Economic Behavior.* Princeton.

Decision number	DE (4)	IT (4)	FR (4)	NL (2)	BE (2)	LU (1)	Decision weight W_{EU6}	$W_{EU6} \mid M_{Benelux}$	$W_{EU6} \mid U_{Benelux}$
1	1	1	1	1	1	1	17	17	17
2	1	1	1	1	1	0	16	17	12
3	1	1	1	1	0	1	15	17	12
4	1	1	1	0	1	1	15	17	12
5	1	1	1	1	0	0	14	12	12
6	1	1	1	0	1	0	14	12	12
7	1	1	1	0	0	1	13	12	12
8	1	1	0	1	1	1	13	13	13
9	1	0	1	1	1	1	13	13	13
10	0	1	1	1	1	1	13	13	13
11	1	1	1	0	0	0	12	12	12
12	1	1	0	1	1	0	12	13	-
13	1	0	1	1	1	0	12	13	-
14	0	1	1	1	1	0	12	13	-
15	1	1	0	1	0	1	-	13	-
16	1	1	0	0	1	1	-	13	-
17	1	0	1	1	0	1	-	13	-
18	1	0	1	0	1	1	-	13	-
19	0	1	1	1	0	1	-	13	-
20	0	1	1	0	1	1	-	13	-

Voting rule	Penrose/Banzhaf influence probability						P/B sensitivity	Mean majority deficit	P/B efficiency
W_{EU6}	20	20	20	12	12	0	84	18	14
$W_{EU6} \mid M_{Benelux}$	24	24	24	12	12	12	108	6	20
$W_{EU6} \mid U_{Benelux}$	18	18	18	6	6	6	72	24	11

Exhibit 1: Weighted voting rule of the EEC 1958-1972 and two bloc voting variants. The voting rule W_{EU6} used voting weights 4, 4, 4, 2, 2, 1, and at least 12 voting weights are necessary for a positive decision. The variants that treat Benelux as a bloc decide internally either by majority ($W_{EU6} \mid M_{Benelux}$), or by unanimity ($W_{EU6} \mid U_{Benelux}$). The formation of the bloc may increase (24) or decrease (18) the influence probabilities of the voters that stand by themselves. The influence probabilities may be obtained by counting the critical decisions, or using Theorem 4. All values in the lower table are multiples of 1/64.